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DETERMINATE SYSTEMS

Stability of the Trinomial Linear Difference Equations with Two Delays[1]

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Abstract—For the zero solution of the difference equation $x(n) = ax(n-m) + bx(n-k)$ with arbitrary delays k, m, the formulas of the stability domain boundaries were derived. For different k and m, the stability domains were compared in the quadrants of the plane (a, b).

1. INTRODUCTION

We intend to study for stability the zero solution of the linear difference equation

$$x(n) = ax(n-m) + bx(n-k) \tag{1}$$

with the arbitrary real numbers a and b and two delays $k, m \in \mathbb{N}$. We assume for definiteness that $k > m$. The characteristic polynomial for Eq. (1) is as follows:

$$f(\lambda) = \lambda^k - a\lambda^{k-m} - b. \tag{2}$$

The general problem of studying polynomial (2) for stability was considered in [1, 2]. In extension of [2], we pose the problems of (i) establishing the formulas for the stability domain boundaries of Eq. (1), (ii) determining the cases where, depending on the signs of a and b, the variations in the delays k and m in (1) retain stability, and (iii) discussing, with regard for [2], what happens with the stability domain of (1) if $k \to \infty$, $m \to \infty$.

S.A. Levin and R.B. May [3] proved that satisfaction of the inequality $0 > b > -2\sin\frac{\pi}{2(2k-1)}$ is necessary and sufficient for asymptotic stability of Eq. (1) for $m = 1$, $a = 1$. Various generalizations of Eq. (1) with $m = 1$ were considered in [4, 5]. The problem of stability of (1) for $m = 1$ was solved in [6]. Equation (1) with $m = k-1$ was studied in [7], and Eq. (1) was considered under some dependences between a and b in [8].

2. STABILITY OF EQUATION (1)

By replacing in polynomial (2) the pair of delays (k, m) by (k_1, m_1) and the variable λ by μ, we obtain the polynomial $f_1(\mu) = \mu^{k_1} - a\mu^{k_1-m_1} - b$. If $k = dk_1$ and $m = dm_1$, then for any root λ of the polynomial f the number $\mu = \lambda^d$ is the root of f_1 and *vice versa*. Consequently, the asymptotic stability domains for the pairs of delays k and m and k_1 and m_1 coincide. Therefore, it is the delays k and m that without loss of generality may be conveniently regarded as coprimes. The following well-known property of the simple natural coprime numbers is formulated as a theorem.

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Theorem 1. *Let k and m be coprimes and $k > m$. Then, there exists a pair of natural numbers (j, s) such that*

$$|mj - ks| = 1, \quad j < k, \quad s \text{ odd}. \tag{3}$$

If m is odd, then this pair is unique; if m is even, then there are precisely two such pairs in one of which j is even and in the other, odd.

The next theorem establishes the boundary of the stability domain of (1) for $k > m$.

Theorem 2. *Let k and m be coprimes and $k > m$. The zero solution of Eq.* (1) *is asymptotically stable if and only if the pair (a, b) is the internal point of the finite domain bounded by the lines*

I. $a + b = 1$,

II. $a = \dfrac{\sin k\omega}{\sin(k-m)\omega}, \quad b = -\dfrac{\sin m\omega}{\sin(k-m)\omega}$,

III. $(-1)^m a + (-1)^k b = 1$,

IV. $a = (-1)^m \dfrac{\sin k\omega}{\sin(k-m)\omega}, \quad b = -(-1)^k \dfrac{\sin m\omega}{\sin(k-m)\omega}$,

where ω varies between $j\pi/k$ and $s\pi/m$. Here, j and s are natural numbers satisfying condition (3).

See the proof in the Appendix.

Note 1. Existence of the numbers j and s with the properties stated in Theorem 2 is provided by Theorem 1. For the odd m, the same theorem guarantees uniqueness. For the even m, any of the two pairs of Theorem 1 (the same in curves II and IV) can be taken as (j, s) in Theorem 2. For some coprimes k and m, we indicate the numbers j and s meeting condition (3).

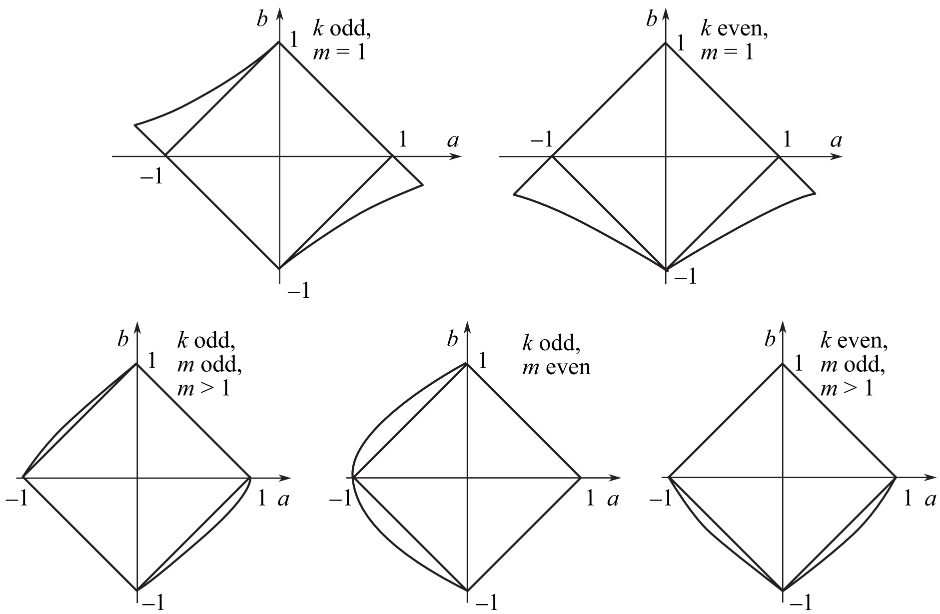
k	2	3	3	3	4	4	5	5	5	5	5	5	6	6	7	7	7	7	7	7	7	7	7
m	1	1	2	2	1	3	1	2	2	3	4	4	1	5	1	2	2	3	4	4	5	6	6
j	1	2	1	2	3	1	4	3	2	2	1	4	5	1	6	3	4	2	5	2	4	1	6
s	1	1	1	1	1	1	1	1	1	1	1	3	1	1	1	1	1	1	3	1	3	1	5.

Note 2. If $mj - ks = 1$ (for example, for $k = 9$, $m = 5$, $j = 2$, $s = 1$), then in the text of Theorem 2 the variable ω runs the interval $(s\pi/m, j\pi/k)$; if $mj - ks = -1$ (for example, for $k = 7$, $m = 5$, $j = 4$, $s = 3$, and also for any k, provided that $m = 1$), then the interval is $(j\pi/k, s\pi/m)$. In both cases, the length of the interval is $\pi/(km)$.

Note 3. We draw attention to the difference between the cases of $m > 1$ and $m = 1$. For any $m > 1$, curve II of Theorem 2 connects the point $a = 0$, $b = (-1)^j$ with the point $a = -1$, $b = 0$ (see the figure); curve IV is symmetrical to curve II. If $m = 1$, then curve II connects the point $a = 0$, $b = -(-1)^k$ with the point $a = -k/(k-1)$, $b = -(-1)^k/(k-1)$; curve IV also is symmetrical to curve II.

Definition 1. We denote by $D(k, m)$ the set of pairs (a, b) such that Eq. (1) with the given coefficients a and b and delays k and m is asymptotically stable.

Let us compare in detail the asymptotic stability domains in the quadrants of the plane (a, b) for various values of k and m. In what follows, $A \supset B$ means that the set B is the proper subset of the set A.



Domains of asymptotic stability of Eq. (1); k and m are coprimes; $k > m$. The stability domain $|a| + |b| < 1$ which is common to all k and m is singled out.

Definition 2. We say that $(k_1, m_1) \succ (k_2, m_2)$ in the quadrant $Q_{rt} = \{(a, b) : (-1)^r a \geqslant 0, (-1)^t b \geqslant 0\}$ $(r, t = 0, 1)$ if $D(k_1, m_1) \bigcap Q_{rt} \supset D(k_2, m_2) \bigcap Q_{rt}$. We say that $(k_1, m_1) \approx (k_2, m_2)$ in Q_{rt} if $D(k_1, m_1) \bigcap Q_{rt} = D(k_2, m_2) \bigcap Q_{rt}$. We say that the pairs (k_1, m_1) and (k_2, m_2) are incomparable in Q_{rt} if the trichotomy

$$(k_1, m_1) \succ (k_2, m_2) \quad \text{or} \quad (k_2, m_2) \succ (k_1, m_1) \quad \text{or} \quad (k_1, m_1) \approx (k_2, m_2) \tag{4}$$

is not true in Q_{rt}. Obviously, $(kd, md) \approx (k, m)$ is satisfied in any quadrant. In any quadrant, the relation $\succ$ and the equivalence relation $\approx$ are coordinated: if $(k_1, m_1) \approx (k_2, m_2)$ and $(k_3, m_3) \approx (k_4, m_4)$ and $(k_1, m_1) \succ (k_3, m_3)$, then $(k_2, m_2) \succ (k_4, m_4)$. The following theorem allows one either to determine for any pairs of delays the occurring term of trichotomy (4) or to state that the pairs are incomparable.

Theorem 3. *Let (k_1, m_1), (k_2, m_2), (k_3, m_3), and (k_4, m_4) be four pairs of natural coprimes, and let $k_r > m_r$ $(1 \leqslant r \leqslant 4)$.*

(1) *Then, $(k_1, m_1) \approx (k_2, m_2)$ in Q_{00}.*

(2) *Let k_1 and k_2 be odd and k_3 and k_4 be even.*

(2.1) *If $k_1 < k_2$, $m_1 \leqslant m_2$ or $k_1 \leqslant k_2$, $m_1 < m_2$, then $(k_1, m_1) \succ (k_2, m_2) \succ (k_3, m_3) \approx (k_4, m_4)$ in Q_{10}.*

(2.2) *If $k_1 > k_2$, $m_1 < m_2$, then (k_1, m_1) and (k_2, m_2) are incomparable in Q_{10}.*

(3) *Let $k_1 + m_1$ and $k_2 + m_2$ be odd and $k_3 + m_3$ and $k_4 + m_4$ be even.*

(3.1) *If $k_1 < k_2$, $m_1 \leqslant m_2$ or $k_1 \leqslant k_2$, $m_1 < m_2$, then $(k_1, m_1) \succ (k_2, m_2) \succ (k_3, m_3) \approx (k_4, m_4)$ in Q_{11}.*

(3.2) *If $k_1 > k_2$, $m_1 < m_2$, then (k_1, m_1) and (k_2, m_2) are incomparable in Q_{11}.*

(4) *Let m_1 and m_2 be odd and m_3 and m_4 be even.*

(4.1) *If $k_1 < k_2$, $m_1 \leqslant m_2$ or $k_1 \leqslant k_2$, $m_1 < m_2$, then $(k_1, m_1) \succ (k_2, m_2) \succ (k_3, m_3) \approx (k_4, m_4)$ in Q_{01}.*

(4.2) *If $k_1 > k_2$, $m_1 < m_2$, then (k_1, m_1) and (k_2, m_2) are incomparable in Q_{01}.*

3. EXAMPLES AND COMMENTS

Theorem 2 does not contradict the description of the stability domain of the characteristic Eq. (2) in [2]. It also does not contradict the results of studying Eq. (1) in [3] with $a = 1$, $m = 1$, [6] with $m = 1$, [7] with $m = k-1$, and [8] with arbitrary k and m, but interdependent a and b. We present examples for Theorem 3. In Example 1 we take into consideration the fact that for $k = m$ the asymptotic stability domain of (1) is the band $|a+b| < 1$. Therefore, the domain $D(k,k)$ equal to the domain $D(1,1)$ is maximum in Q_{10} and Q_{01} and minimum in Q_{11}. In Q_{00} all domains are the same.

Example 1. We construct relations $\succ$ and $\approx$ in the quadrants Q_{rt} $(r,t = 0,1)$ in the set of delay pairs $(k,m) \in \{(1,1),(6,5),(7,4)\}$. Theorem 3 provides the following results. In Q_{00}, $(1,1) \approx (6,5) \approx (7,4)$; in Q_{10}, $(1,1) \succ (7,4) \succ (6,5)$; in Q_{11}, the pairs $(6,5)$ and $(7,4)$ are incomparable, $(6,5) \succ (1,1)$, $(7,4) \succ (1,1)$; in Q_{01} $(1,1) \succ (6,5) \succ (7,4)$.

Example 2. We assume that in (1) we can control the delays by choosing $82 \leqslant k \leqslant 89$, $63 \leqslant m \leqslant 69$. For each quadrant, we take those pairs (k,m) which maximize the asymptotic stability domains in it. To this end, each pair (k,m) from the prescribed range is reduced by the common divisors. Theorem 3 provides the following results. In Q_{00}, all domains are identical; in Q_{10}, the maximum stability domain for $(k,m) = (85,68) \approx (5,4)$; in Q_{11}, as well as in Q_{01}, for $(k,m) = (84,63) \approx (4,3)$.

According to Theorem 3, reduction in delays, generally speaking, is beneficial for stability. But common divisors of k and m are more important, and the greater the common divisor, the better, in principle, for stability. Nevertheless, this statement is not universal because one has to take into account the quadrant where the coefficients a and b are located. For Example 2, within the limits of delays in the quadrant Q_{10}, the choice of $(k,m) = (84,63) \approx (4,3)$ gives no gain in the stability domain because k is even. But the choice of the pair $(k,m) = (85,68) \approx (5,4)$ extends to advantage the stability domain in Q_{10} despite the fact that the delays in the pair (5,4) are greater than the corresponding delays in the pair (4,3).

S.A. Kuruklis [6] considered stability of Eq. (1) for $m = 1$:

$$x(n) = ax(n-1) + bx(n-k). \tag{5}$$

Theorem 4 ([6], Theorem 4). *Let a and b be arbitrary real numbers and k be a natural number, $k > 1$. The zero solution of Eq.* (5) *is asymptotically stable if and only if* $|a| < \dfrac{k}{k-1}$ *and*

$$-\sqrt{a^2+1-2|a|\cos\omega} < b < 1-|a| \quad \textit{for even} \quad k,$$

$$|b+a| < 1 \quad \textit{and} \quad |b| < \sqrt{a^2+1-2|a|\cos\omega} \quad \textit{for odd} \quad k,$$

where ω is the solution of the equation $\dfrac{\sin(k-1)\omega}{\sin k\omega} = \dfrac{1}{|a|}$ *within the interval* $(0,\pi/k)$.

It follows from the aforementioned that for $k \to \infty$ in Eq. (5) the stability domain decreases and approaches the "Cohn domain" $|a|+|b| < 1$ [9]. The tendency of the stability domain of polynomial (2) to the "Cohn domain" for $k \to \infty$ is also mentioned in [2]. This is true if, for example, m is constant or assumes values that are coprimes with k. Let us assume in (2) that $m = k/2$ and k have even values. Then, for any even k the stability domain of the polynomial (2) is bounded by the triangular with the sides $b = -1$, $b+a = 1$, and $b-a = 1$ [10], thus appreciably exceeding the bounds of the "Cohn domain."

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APPENDIX

The proof of Theorem 2 is preceded by some lemmas.

Lemma 1. *If k and m are coprimes, $k > m > 1$, and $a \leqslant -1$, then the zero solution of Eq.* (1) *is not asymptotically stable.*

Proof. To prepare the use of the principle of argument, we assume in (2) that $\lambda = \exp(i\omega)$ and obtain

$$u(\omega) = \operatorname{Re} f(e^{i\omega}) = \cos k\omega - a\cos(k-m)\omega - b, \tag{A.1}$$

$$v(\omega) = \operatorname{Im} f(e^{i\omega}) = \sin k\omega - a\sin(k-m)\omega. \tag{A.2}$$

First, we show that for $a \leqslant -k/(k-m)$ Eq. (1) is unstable. We introduce the function $g(\omega) = \dfrac{\sin(k-m)\omega}{\sin k\omega}$. Owing to the coprimality of k and m, the equations $v(\omega) = 0$ and $g(\omega) = 1/a$ are equivalent within the intervals $(0,\pi)$ and $(\pi, 2\pi)$. For all a, for some deleted neighborhood of zero the equation $g(\omega) = 1/a$ has $2k-2$ roots over the interval $[0, 2\pi)$, and, with regard for the roots $\omega = 0$ and $\omega = \pi$, the equation $v(\omega) = 0$ has $2k$ roots over the same interval. The function g satisfies the differential equation $g''_{\omega\omega} = m(2k-m)g - 2k(\operatorname{ctg} k\omega)g'_\omega$; therefore, for $g'_\omega = 0$ the sign of $g''_{\omega\omega}$ coincides with the sign of $g(\omega)$. Consequently, over each interval $(j\pi/k, (j+1)\pi/k)$ $(0 \leqslant j \leqslant 2k-1)$ the function g is either monotone, or unimodal with a positive minimum, or unimodal with a negative minimum. Therefore, as a varies from 0 to $-\infty$, the equation $g(\omega) = 1/a$ can only lose the roots when $(1/a)$ reaches the maximum of the function g.

Case 1 (m is even). Then, k is odd, and we get for $\omega = \pi/2$ that $g(\omega) = -1$, $g'_\omega = 0$, $g''_{\omega\omega} < 0$. Consequently, $\omega = \pi/2$ is the point of negative maximum of g; therefore, for $a \leqslant -1$ the equation $g(\omega) = 1/a$ loses a root. The locus (A.1), (A.2) has less than $2k$ intersections with the axis $v = 0$ for $\omega \in [0, 2\pi)$, the increment of the locus is smaller than $2k\pi$, Eq. (1) is unstable.

Case 2 (m is odd). Then, over the interval $((k-1)\pi/k, \pi)$ the function $g(\omega)$ grows monotonically from $(-\infty)$ to $-(k-m)/k$. Consequently, for $a \leqslant -k/(k-m)$ the equation $g(\omega) = 1/a$ loses a root over this interval. Therefore, the locus (A.1), (A.2) has less than $2k$ points of intersection with the axis $v = 0$, which makes Eq. (1) unstable. Therefore, for $a \leqslant -k/(k-m)$ Eq. (1) is unstable.

Let $-k/(k-m) \leqslant a \leqslant -1$. We consider the line

$$v\cos\alpha - u\sin\alpha = 0 \tag{A.3}$$

on the plane (u, v); the value of α will be chosen in what follows. The points of intersection of the locus (A.1), (A.2) with (A.3) are the zeros of the function

$$\varphi(\omega) = \cos(k\omega + \alpha) - a\cos((k-m)\omega + \alpha) - b\cos\alpha. \tag{A.4}$$

For the locus (A.1),(A.2) to pass $2k$ times the origin, it is necessary that for $\omega \in [0, 2\pi)$ the function $\varphi(\omega)$ have at least k positive maxima and k negative minima. But its derivative

$$\varphi'_\omega = -k\sin(k\omega + \alpha) + a\sin((k-m)\omega + \alpha) \tag{A.5}$$

has at most $2k$ roots over the same interval of values of ω ([8], Lemma 3). Therefore, it is necessary for stability that all maxima of the function φ be positive and all minima, negative. We use Theorem 1 to determine the natural numbers j and s meeting condition (3). Let us assume that $\omega = s\pi/m$. We assume in (A.3)–(A.5) that $\alpha = \pi/m$ if $mj-ks = 1$ and $\alpha = -\pi/m$ if $mj-ks = -1$. In both cases,

$$k\omega + \alpha = (mj-1)\pi/k + \alpha = j\pi; \quad (k-m)\omega + \alpha = (j-s)\pi. \tag{A.6}$$

We obtain from (A.4)–(A.6) that

$$\varphi = (-1)^j(1+a) - b\cos\alpha, \quad \varphi'_\omega = 0, \quad \varphi''_{\omega\omega} = (-1)^j(-k^2 - a(k-m)^2). \tag{A.7}$$

Case 1 (m is odd). Then, since $|mj-ks| = 1$, we get that $(-1)^j = (-1)^{k+1}$. Along with this, we deduce from (2) the equalities for the characteristic polynomial f of (1):

$$f(-1) = (-1)^k(1+a) - b, \quad f(-\infty) = (-1)^k(+\infty). \tag{A.8}$$

Case 1.1 (k odd). If $b \geqslant 0$, then we get from (A.7), equality $(-1)^j = (-1)^{k+1}$, inequality $|\alpha| < \pi/2$ (because $m > 1$), and $a \geqslant -k/(k-m) > -k^2/(k-m)^2$ that $\varphi \leqslant 0$, $\varphi'_\omega = 0$, and $\varphi''_{\omega\omega} < 0$. Consequently, φ has a nonpositive maximum, which is contradictory to asymptotic stability. If $b \leqslant 0$, then in view of inequality $a \leqslant -1$ we obtain from (A.8) that f has a real root $\lambda \leqslant -1$, which is incompatible with asymptotic stability.

Case 1.2 (k is even). If $b \leqslant 0$, then from (A.7), equality $(-1)^j = (-1)^{k+1}$, and the same inequalities in α and a as in Case 1.1 we obtain $\varphi \geqslant 0$, $\varphi'_\omega = 0$, $\varphi''_{\omega\omega} > 0$. Consequently, φ has a nonnegative minimum, which contradicts asymptotic stability. If $b \geqslant 0$, then we obtain from (A.8) that the characteristic polynomial f has the root $\lambda \leqslant -1$ and Eq. (1) is not asymptotically stable.

Case 2 (m is even). Then, by Theorem 1 one can determine two pairs (j, s) satisfying (3), j being even in one pair and odd in the other. If $b \geqslant 0$, then we take an even j and its corresponding odd s. For $\omega = s\pi/m$, we get from (A.7) that $\varphi \leqslant 0$, $\varphi'_\omega = 0$, $\varphi''_{\omega\omega} < 0$, which contradicts stability. If $b \leqslant 0$, then we take an odd j and its corresponding odd s. For $\omega = s\pi/m$, we get from (A.7) that $\varphi \geqslant 0$, $\varphi'_\omega = 0$, $\varphi''_{\omega\omega} > 0$, which contradicts asymptotic stability and proves Lemma 1.

Lemma 2. *Let k and m be coprimes and $k > m$. Let for any $a \in (-1, 0)$ $\omega_j(a)$ $(j = 1, 2, \ldots, k-1)$ be the root of the equation*

$$\sin k\omega - a\sin(k-m)\omega = 0 \tag{A.9}$$

within the interval $\left(-\dfrac{\pi}{2k} + \dfrac{j\pi}{k}, \dfrac{j\pi}{k} + \dfrac{\pi}{2k}\right)$. *Let also $\omega_0(a) \equiv 0$, $\omega_k(a) \equiv \pi$. Let for each j $(j = 1, 2, \ldots, k-1)$ the natural number s_j and the integer p_j be constructed such that*

$$mj - ks_j = p_j, \quad s_j \textit{ odd}, \quad |p_j| < k. \tag{A.10}$$

Then, for any j $(j = 1, 2, \ldots, k-1)$:

(1) *The function $\omega_j(a)$ is definite and continuous for any $a \in (-1, 0)$.*

(2) *The function $\omega_j(a)$ features the following properties of monotonicity, and the domains of the range space of the functions $\omega_j(a)$, $k\omega_j(a)$, $m\omega_j(a)$, $(k-m)\omega_j(a)$ obey the following inequalities:*

(2.1) *If $p_j < 0$ and $|p_j| \leqslant m/2$, then $\omega_j(a)$ decreases monotonically and*

$$\frac{j\pi}{k} < \omega_j(a) < \frac{j\pi}{k} - \frac{p_j\pi}{km} = \frac{s_j\pi}{m}; \quad j\pi < k\omega_j(a) < j\pi - \frac{p_j\pi}{m}; \tag{A.11}$$
$$s_j\pi + \frac{p_j\pi}{k} < m\omega_j(a) < s_j\pi;$$
$$(j-s_j)\pi - \frac{p_j\pi}{k} < (k-m)\omega_j(a) < (j-s_j)\pi - \frac{p_j\pi}{m}.$$

(2.2) *If* $p_j > 0$ *and* $|p_j| \leqslant m/2$, *then* $\omega_j(a)$ *increases monotonically and*

$$\frac{s_j\pi}{m} = \frac{j\pi}{k} - \frac{p_j\pi}{km} < \omega_j(a) < \frac{j\pi}{k}; \quad j\pi - \frac{p_j\pi}{m} < k\omega_j(a) < j\pi; \tag{A.12}$$
$$s_j\pi < m\omega_j(a) < s_j\pi + \frac{p_j\pi}{k};$$
$$(j - s_j)\pi - \frac{p_j\pi}{m} < (k-m)\omega_j(a) < (j - s_j)\pi - \frac{p_j\pi}{k}.$$

(2.3) *If* $p_j < 0$ *and* $|p_j| > m/2$, *then* $\omega_j(a)$ *decreases monotonically and*

$$\frac{j\pi}{k} < \omega_j(a) < \frac{j\pi}{k} + \frac{(k+p_j)\pi}{(2k-m)k} = \frac{(2j+1-s_j)\pi}{2k-m}; \tag{A.13}$$
$$j\pi < k\omega_j(a) < j\pi + \frac{(k+p_j)\pi}{2k-m};$$
$$s_j\pi + \frac{p_j\pi}{k} < m\omega_j(a) < s_j\pi + \frac{(2p_j+m)\pi}{2k-m};$$
$$(j - s_j)\pi - \frac{p_j\pi}{k} < (k-m)\omega_j(a) < (j - s_j)\pi + \frac{(k-m-p_j)\pi}{2k-m}.$$

(2.4) *If* $p_j > 0$ *and* $|p_j| > m/2$, *then* $\omega_j(a)$ *increases monotonically and*

$$\frac{(2j-1-s_j)\pi}{2k-m} = \frac{j\pi}{k} - \frac{(k-p_j)\pi}{(2k-m)k} < \omega_j(a) < \frac{j\pi}{k}; \tag{A.14}$$
$$j\pi - \frac{(k-p_j)\pi}{2k-m} < k\omega_j(a) < j\pi;$$
$$s_j\pi + \frac{(2p_j-m)\pi}{2k-m} < m\omega_j(a) < s_j\pi + \frac{p_j\pi}{k};$$
$$(j - s_j)\pi - \frac{(k-m+p_j)\pi}{2k-m} < (k-m)\omega_j(a) < (j - s_j)\pi - \frac{p_j\pi}{k}.$$

Proof. We introduce the function $h(\omega) = \dfrac{\sin k\omega}{\sin(k-m)\omega}$ and consider Case 1: $p_j < 0$, $|p_j| \leqslant m/2$. Then, $h(j\pi/k) = 0$ and $h(s_j\pi/m) = -1$ according to (A.10). Additionally, $\sin(k-m)\omega$ does not vanish for $\omega \in (j\pi/k, s_j\pi/m)$; therefore, h is continuous over this interval and, consequently, assumes all values between 0 and (-1). From (A.10) and the inequality $|p_j| \leqslant m/2$, we obtain $s_j/m - j/k = -p_j/(mk) \leqslant 1/(2k)$; therefore, $j\pi/k - \pi/(2k) < j\pi/k < s_j\pi/m \leqslant j\pi/k + \pi/(2k)$. Thus, the equation $a = h(\omega)$ that is equivalent to (A.9) in $(0, \pi)$ is solvable for any $a \in (-1, 0)$ within the interval $(-\pi/(2k) + j\pi/k, j\pi/k + \pi/(2k))$ and even within a narrower interval $(j\pi/k, s_j\pi/m)$.

Similarly, in Case 2 (A.9) is solvable, $p_j > 0$, $|p_j| \leqslant m/2$; in Case 3, $p_j < 0$, $|p_j| > m/2$; and Case 4, $p_j > 0$, $|p_j| > m/2$. In the nonintersecting neighborhoods of the points $j\pi/k$, $j = 1, 2, \ldots, k-1$, we count in all $(k-1)$ roots of Eq. (A.9). By virtue of symmetry and with regard for the roots $\omega_0 = 0$ and $\omega_k = 0$, in $[0, 2\pi)$ Eq. (A.9) has $2k$ roots in all. We return to Case 1. If h was not monotone over the interval $(j\pi/k, s_j\pi/m)$, then the equation $h(\omega) = a$ and, consequently, (A.9), would have for some $a \in (-1, 0)$ at least two roots over this interval, and the number of roots of (A.9) over $[0, 2\pi)$ would exceed $2k$, which contradicts Lemma 3 [8]. Therefore, h is monotone over $(j\pi/k, s_j\pi/m)$. Of course, h decreases. Consequently, for h there exists the continuous inverse function $\omega_j(a) : (-1, 0) \to (j\pi/k, s_j\pi/m)$ which decreases monotonically. The remaining inequalities in (A.11) are obtained from the inequality $j\pi/k < \omega_j(a) < s_j\pi/m$ by direct count with the use of (A.10). Item 1 in Case 1 and Item 2.1 are proved. Item 1 in Cases 2, 3, and 4 and Items 2.2, 2.3, and 2.4 are proved in a similar manner, which proves Lemma 2.

Lemma 3. *Let k and m be coprimes $k > m$, $a \in (-1, 0)$ and $\omega_j(a)$ $(j = 0, 1, \ldots, k)$ be the functions defined in Lemma 2. Let u^*, v be the locus* (A.1), (A.2) *for* $b = 0$:

$$u^*(\omega) = \cos k\omega - a\cos(k-m)\omega; \quad v(\omega) = \sin k\omega - a\sin(k-m)\omega. \tag{A.15}$$

Then, for all $a \in (-1, 0)$

$$v(\omega_j(a)) = 0;\ u^*(\omega_j(a)) > 0,\ \text{if } j \text{ is even};\quad u^*(\omega_j(a)) < 0,\ \text{if } j \text{ is odd}; \tag{A.16}$$

$$v'_\omega(\omega_j(a)) > 0,\ \text{if } j \text{ is even};\quad v'_\omega(\omega_j(a)) < 0,\ \text{if } j \text{ is odd}. \tag{A.17}$$

Proof. Comparison of (A.15) with the definition of $\omega_j(a)$ in Lemma 2 indicates that $v(\omega_j(a)) = 0$ exists for any $j = 0, 1, \ldots, k$. The equality $m\omega_j(a) = s_j\pi + \delta_j(a)$, where s_j and p_j satisfy (A.10), $\operatorname{sgn}\delta_j(a) = \operatorname{sgn} p_j$, $|\delta_j(a)| < \pi$, is obtained from (A.11)–(A.14) for any $j = 1, 2, \ldots, k-1$. We also deduce from this the equality $(k-m)\omega_j(a) = (j - s_j)\pi + \gamma_j(a)$, where $\operatorname{sgn}\gamma_j(a) = -\operatorname{sgn} p_j$, $|\gamma_j(a)| < \pi$. Therefore, with regard for (A.10) and (A.9) we obtain from (A.15) that

$$\begin{aligned} u^*(\omega_j(a)) &= \cos k\omega_j(a) - \frac{\sin k\omega_j(a)}{\sin(k-m)\omega_j(a)}\cos(k-m)\omega_j(a) \\ &= -\frac{\sin m\omega_j(a)}{\sin(k-m)\omega_j(a)} = -\frac{\sin|\delta_j(a)|\operatorname{sgn} p_j}{(-1)^j \sin|\gamma_j(a)|(-\operatorname{sgn} p_j)}. \end{aligned}$$

From this we get (A.16). It follows from (A.15) that

$$v'_\omega = (k-m)u^*(\omega) + m\cos k\omega. \tag{A.18}$$

By virtue of (A.11)–(A.14), $k\omega_j(a) = j\pi + \alpha_j(a)$, where $|\alpha_j(a)| \leqslant \pi/2$; therefore, we obtain from (A.18) that $v'_\omega(\omega_j(a)) = (k-m)u^*(\omega_j(a)) + m(-1)^j\cos\alpha_j(a)$, whence (A.17) follows for $j = 1, 2, \ldots, k-1$. For $\omega_0(a) \equiv 0$ and $\omega_k(a) \equiv \pi$, the conclusion of the lemma is verified by simple count, which proves Lemma 3.

Lemma 4. *Let k and m be coprimes and $k > m$, let $\omega_j(a)$ be the functions defined in Lemma 2; let $mj - ks_j = p_j$, $mr - ks_r = p_r$, where s_j, s_r are odd natural numbers, $|p_j| \leqslant k$, and $|p_r| \leqslant k$; let*

$$\beta_j(a) = |(k-m)\omega_j(a) - (j - s_j)\pi|, \tag{A.19}$$

$j, r = 0, 1, \ldots, k$. *For any $a \in (-1, 0)$, if $|p_j| < |p_r|$, then $\beta_j(a) < \beta_r(a)$.*

Proof. We first assume that $0 < |p_j| < |p_r| < k$. It follows from (A.11)–(A.14) and (A.19) that $\beta_j(a) = (-\operatorname{sgn} p_j)((k-m)\omega_j(a) - (j - s_j)\pi)$. Hence, owing to (A.10) we obtain

$$\begin{aligned} k\omega_j(a) &= (-\operatorname{sgn} p_j)\frac{k\beta_j(a) - |p_j|\pi}{k-m}; \\ (k-m)\omega_j(a) &= (-\operatorname{sgn} p_j)\beta_j(a) + (j - s_j)\pi. \end{aligned} \tag{A.20}$$

From (A.9) and (A.20) we obtain $a = -\sin\dfrac{k\beta_j(a) - |p_j|\pi}{k-m}\Big/\sin\beta_j(a)$. Therefore, the functions $\beta_j(a)$ and $\beta_r(a)$ are inverse, respectively, to the functions

$$a_j(\beta) = -\frac{\sin\frac{k\beta - |p_j|\pi}{k-m}}{\sin\beta}, \quad a_r(\beta) = -\frac{\sin\frac{k\beta - |p_r|\pi}{k-m}}{\sin\beta}. \tag{A.21}$$

From (A.19) and (A.11)–(A.14), we obtain the limits of variation of β in $a_j(\beta)$:

$$\frac{|p_j|\pi}{k} < \beta < \frac{|p_j|\pi}{m}, \quad \text{if} \quad |p_j| \leqslant \frac{m}{2}; \qquad \text{(A.22)}$$
$$\frac{|p_j|\pi}{k} < \beta < \frac{(k-m+|p_j|)\pi}{2k-m}, \quad \text{if} \quad |p_j| > \frac{m}{2}.$$

Similarly, in the function $a_r(\beta)$

$$\frac{|p_r|\pi}{k} < \beta < \frac{|p_r|\pi}{m}, \quad \text{if} \quad |p_r| \leqslant \frac{m}{2}; \qquad \text{(A.23)}$$
$$\frac{|p_r|\pi}{k} < \beta < \frac{(k-m+|p_r|)\pi}{2k-m}, \quad \text{if} \quad |p_r| > \frac{m}{2}.$$

For $|p_j| < |p_r|$, the lower boundary of $\beta_j(a)$ in (A.22) is smaller than the lower boundary of $\beta_r(a)$ in (A.23). The same is true for the upper boundaries. The range spaces of $\beta_j(a)$ and $\beta_r(a)$ need not intersect: for example, for $k = 9$, $m = 5$, $j = 2$, $r = 6$ we determine $p_j = 1$, $p_r = 3$ from (A.10), and by virtue of (A.11)–(A.14), (A.19) we obtain $\pi/9 < \beta_j(a) < \pi/5 < \pi/3 < \beta_r(a) < 7\pi/13$. But these domains may be overlapping: for example, for $k = 13$, $m = 3$, $j = 4$, $r = 12$ we obtain $p_j = -1$, $p_r = -3$ and $\pi/13 < \beta_j(a) < \pi/3$, $3\pi/13 < \beta_r(a) < 13\pi/23$; the interval $(3\pi/13, \pi/3)$ is common to both domains.

Let us consider both functions $\beta_j(a)$ and $\beta_r(a)$ over the common part of their domains (if the common part is empty, the conclusion of the lemma is trivial; outside the common part the inequality $\beta_j(a) < \beta_r(a)$ is obvious). We first note that if $|p_j| \leqslant m/2$, then according to (A.22)

$$0 < \frac{k\beta_j(a) - |p_j|\pi}{k-m} < \frac{k\frac{p_j\pi}{m} - |p_j|\pi}{k-m} \leqslant \frac{\pi}{2};$$

if $|p_j| > m/2$, then according to (A.22)

$$0 < \frac{k\beta_r(a) - |p_r|\pi}{k-m} < \frac{k\frac{(k-m+|p_j|)\pi}{2k-m} - |p_j|\pi}{k-m} = \frac{(k-|p_j|)\pi}{2k-m} < \frac{\pi}{2}.$$

Similarly, according to (A.23), $0 < (k\beta_r(a) - |p_r|\pi)/(k-m) < \pi/2$. Therefore, for $|p_j| < |p_r|$ (see (A.21)) we get $(-1) < a_j(\beta) < a_r(\beta) < 0$. Hence, since the inverse functions $\beta_j(a)$ and $\beta_r(a)$ are monotonically decreasing, we obtain the inequality $\beta_j(a) < \beta_r(a)$.

It remains to consider the cases of $0 = p_j$ and $|p_r| = k$. Let $0 = |p_j| < |p_r|$. Then, m is certainly odd, $j = k$, and $s_j = m$ by virtue of (A.10). Since $\omega_k(a) \equiv \pi$, we obtain according to (A.19) that $0 = \beta_k(a) = \beta_j(a) < \beta_r(a)$.

Let now $|p_r| = k$.

Case 1 (m is odd). We obtain from $|mr - ks_r| = k$ that $r = 0$ and $s_r = 1$. Therefore, it follows from (A.19) and the equality $\omega_0(a) \equiv 0$ that $\beta_j(a) < \beta_r(a) = \beta_0(a) = \pi$.

Case 2 (m is even). The k is odd and two subcases are possible.

Case 2.1 ($r = 0$, $s_r = 1$). Then, owing to the equality $\omega_0(a) \equiv 0$, we obtain $\beta_j(a) < \beta_r(a) = \beta_0(a) = \pi$ from (A.19).

Case 2.2 ($r = k$, $s_r = m - 1$). Then, owing to the equality $\omega_k(a) \equiv \pi$, we obtain also $\beta_j(a) < \beta_r(a) = \beta_k(a) = \pi$ from (A.19), which proves Lemma 4.

Lemma 5. *Let k and m be coprimes and $k > m$; let $\omega_j(a)$ be the functions defined in Lemma 2; and let the equalities* (A.10) *be constructed for $j = 0, 1, \ldots, k$. For the locus* (A.15), *under any $a \in (-1, 0)$ and $j, r \in \{0, 1, \ldots, k\}$:*

(1) *if j is even, r is even, and $|p_j| < |p_r|$, then $0 < u^*(\omega_j(a)) < u^*(\omega_r(a))$;*

(2) *if j is odd, r is odd, and $|p_j| < |p_r|$, then $0 > u^*(\omega_j(a)) > u^*(\omega_r(a))$.*

Proof. According to Lemma 4, we obtain from the inequality $|p_j| < |p_r|$ that $0 \leqslant \beta_j(a) < \beta_r(a) \leqslant \pi$, and hence

$$\cos\beta_j(a) > \cos\beta_r(a). \tag{A.24}$$

For any $t = 0, 1, \dots, k$ by virtue of (A.11)–(A.14)

$$|k\omega_t(a) - t\pi| \leqslant \pi/2. \tag{A.25}$$

Let j be even and r be even. Then, according to (A.25) $\cos k\omega_t(a) \geqslant 0$ for $t = j$ and $t = r$. Consequently, for $t = j$ and $t = r$ we get from (A.15) and the definition of $\omega_j(a)$ in Lemma 2 that $\cos k\omega_t(a) = \sqrt{1 - a^2\sin^2(k-m)\omega_t(a)}$. Hence (see (A.15)), for $t = j$ and $t = r$

$$u^*(\omega_t(a)) = \sqrt{1 - a^2 + a^2\cos^2(k-m)\omega_t(a)} - a\cos(k-m)\omega_t(a). \tag{A.26}$$

Then, using the definition of $\beta_j(a)$ and (A.19), from evenness of t and oddity of s_t we obtain for $t = j$ and $t = r$ that

$$\cos(k-m)\omega_t(a) = -\cos\beta_t(a). \tag{A.27}$$

Therefore, we deduce from (A.24) that

$$\cos(k-m)\omega_j(a) < \cos(k-m)\omega_r(a). \tag{A.28}$$

Meanwhile, $y(x) = \sqrt{1 - a^2 + a^2x^2} - ax$ is a strictly monotone increasing positive function over the interval $[-1, 1]$ of variation of the variable x for any $a \in (-1, 0)$. Consequently, we obtain from (A.26) and (A.28) that $0 < u^*(\omega_j(a)) < u^*(\omega_r(a))$, which is what we set out to prove.

Let j be odd and r be odd. For $t = j$ and $t = r$, $\cos k\omega_t(a) \leqslant 0$ according to (A.25). Therefore, for $t = j$ and $t = r$, we obtain similar to equality (A.26) that

$$u^*(\omega_t(a)) = -\sqrt{1 - a^2 + a^2\cos^2(k-m)\omega_t(a)} - a\cos(k-m)\omega_t(a). \tag{A.29}$$

By using the definition of $\beta_j(a)$ and (A.19), from oddity of t and oddity of s_t we obtain for $t = j$ and $t = r$ that $\cos(k-m)\omega_t(a) = \cos\beta_t(a)$. Therefore, we deduce from (A.24) that

$$\cos(k-m)\omega_j(a) > \cos(k-m)\omega_r(a). \tag{A.30}$$

Yet, $y(x) = -\sqrt{1 - a^2 + a^2x^2} - ax$ is a strictly monotone increasing negative function over the interval $[-1, 1]$ of variation of the variable x for any $a \in (-1, 0)$. Consequently, we obtain from (A.29) and (A.30) that $0 > u^*(\omega_j(a)) > u^*(\omega_r(a))$, which proves Lemma 5.

Lemma 6. *Let k and m be coprimes and $k > m$. Then (see Definition 1), if $(a, b) \in D(k, m)$, then $((-1)^m a, (-1)^k b) \in D(k, m)$.*

Proof. It suffices to make sure that for the characteristic polynomial $f(a, b, \lambda) = \lambda^k - a\lambda^{k-m} - b$ of Eq. (1) the identity $f((-1)^m a, (-1)^k b, -\lambda) = (-1)^k f(a, b, \lambda)$ holds for the coprimes k and m, which proves Lemma 6.

Proof of Theorem 2. Let k and m be coprimes and $k > m$.

Case 1 ($m > 1$, $a < 0$). Since there is no asymptotic stability for $a \leqslant -1$ (Lemma 1), we seek the boundary of the stability domain in the band $-1 < a < 0$. According to Lemma 3, the locus (A.15) has an increment of the argument $k\pi$ with variation of ω from 0 to π, which entails asymptotic stability of Eq. (1) for $b = 0$. Let c and d be, respectively, the least positive and the greatest negative abscissas of the intersection point of locus (A.15) with the axis $v = 0$. With variation of b, locus (A.1),(A.2) is obtained by translating locus (A.15) to the right ($b < 0$) or to the left ($b > 0$) as a whole. Therefore, it is necessary and sufficient for asymptotic polynomial that the origin remain between the points $(c - b)$ and $(d - b)$. Therefore, the necessary and sufficient condition for asymptotic polynomial of Eq. (1) is as follows:

$$d < b < c;\ 0 < c = \min_j u^*(\omega_j(a)),\ j \text{ is even};\quad 0 > d = \max_j u^*(\omega_j(a)),\ j \text{ odd}. \tag{A.31}$$

Let us determine c, d. Locus (A.15) intersects the axis $v = 0$ for the values of ω, equal to $\omega_0 \equiv 0$, $\omega_j(a)$ $(j = 1, 2, \ldots, k-1)$, $\omega_k(a) \equiv \pi$. According to Lemma 5, in order to determine c, d one needs to assume in (A.10) that $p_j = 0$, $p_j = 1$, and $p_j = -1$.

Case 1.1 (k is odd, m is odd). We assume that $p_j = 0$ in (A.10). By Theorem 1, we obtain a single pair (j, s_j) such that $mj - ks_j = p_j = 0$, $j \leqslant k$, s_j odd. In this pair, $j = k$, $s_j = m$. Since k is odd, $0 > d = u^*(\omega_k(a)) = u^*(\pi) = -1 - a$ (see Lemma 3 and (A.15)). In order to determine c, we assume in (A.10) that $|p_j| = 1$. By Theorem 1, there exists a single pair (j, s_j) such that $|mj - ks_j| = 1$, $j < k$, s is odd. Here, j, of course, is even, and therefore (see Lemma 3 and (A.9))

$$0 < c = u^*(\omega_j(a)) = \cos k\omega_j(a) - a\cos(k-m)\omega_j(a) = -\frac{\sin m\omega_j(a)}{\sin(k-m)\omega_j(a)}. \tag{A.32}$$

By comparing (A.31) with (A.32) and the established boundary $d = -1-a$, we obtain the necessary and sufficient condition for asymptotic stability of Eq. (1) on the half-plane $a < 0$:

$$-1 - a < b < -\sin m\omega_j(a)/\sin(k-m)\omega_j(a), \tag{A.33}$$

where j is such that condition (3) is met. On the half-plane $a < 0$, we establish from (A.33) the boundaries of the stability domain:

$$b = -1 - a \tag{A.34}$$

(line III of Theorem 2) and (see also Lemma 2)

$$b = -\sin m\omega/\sin(k-m)\omega, \quad a = \sin k\omega/\sin(k-m)\omega \tag{A.35}$$

(line II of Theorem 2), where ω varies from $j\pi/k$ to $s_j\pi/m$ (the domain of variation of $\omega_j(a)$ is defined by (A.11), (A.12) in Lemma 2).

Case 1.2 (k is even, m is odd). We assume that $p_j = 0$ in (A.10). By Theorem 1, we obtain a single pair (j, s_j) such that $mj - ks_j = p_j = 0$, $j \leqslant k$, s_j odd. In this pair, $j = k$, $s_j = m$. Since k is even, $0 < c = u^*(\omega_k(a)) = u^*(\pi) = 1 + a$ (see Lemma 3 and (A.15)). In order to determine d, we assume in (A.10) that $|p_j| = 1$. By Theorem 1, there exists a single pair (j, s_j) such that $|mj - ks_j| = 1$, $j < k$, s odd. Obviously, j is odd, and therefore (see Lemma 3 and (A.9))

$$0 > d = u^*(\omega_j(a)) = \cos k\omega_j(a) - a\cos(k-m)\omega_j(a) = -\frac{\sin m\omega_j(a)}{\sin(k-m)\omega_j(a)}. \tag{A.36}$$

By comparing (A.31) with (A.36) and the determined boundary $c = 1 + a$, we establish the necessary and sufficient condition for asymptotic stability of Eq. (1) on the half-plane $a < 0$:

$$1 + a > b > -\sin m\omega_j(a)/\sin(k-m)\omega_j(a), \tag{A.37}$$

where j is such that condition (3) is met. On the half-plane $a < 0$, we obtain from (A.37) the boundaries of the stability domain: $b = 1 + a$ (line III, Theorem 2) and (see also Lemma 3) line (A.35) (line II, Theorem 2) with the intervals of variation of ω as defined by (A.11), (A.12). Formula (A.35) (boundary II, Theorem 2) describes the boundary of the stability domain both in Cases 1.1 and 1.2. Yet, on the boundary (A.35) $b > 0$ in Case 1.1, and in Case 1.2 by formula (A.35) $b < 0$.

Case 1.3 (k is odd, m is even). Obviously, there exists no pair (j, s_j) such that $mj - ks_j = p_j = 0$, $j \leqslant k$, s_j is odd. By Theorem 1, there exist two pairs (j, s_j) such that (A.10) with $|p_j| = 1$ is satisfied. In one of them, j is even, and in the other it is odd. To determine c, we first consider an even j and its corresponding s_j. Then, by Lemma 3 and (A.9) we obtain for c precisely the same formula (A.32). Therefore, the inequality $b < -\sin m\omega_j(a)/\sin(k-m)\omega_j(a)$ will be the necessary and sufficient stability condition in the domain $-1 < a < 0$, $b > 0$. To determine d, we make use of the symmetry mentioned in Lemma 6 and obtain $0 > d = \sin m\omega_j(a)/\sin(k-m)\omega_j(a)$. We have derived an inequality for the stability domain for $a \in (-1, 0)$:

$$\sin m\omega_j(a)/\sin(k-m)\omega_j(a) < b < -\sin m\omega_j(a)/\sin(k-m)\omega_j(a). \tag{A.38}$$

From this formula we establish lines II and IV of Theorem 2 as the boundaries of the stability domain.

For Case 1.3, we assumed until now that an even j and its corresponding s_j were taken. If we take an odd j, then (A.38) will be replaced by

$$\sin m\omega_j(a)/\sin(k-m)\omega_j(a) > b > -\sin m\omega_j(a)/\sin(k-m)\omega_j(a),$$

but the boundaries are defined as before by formulas II and IV of Theorem 2, which corroborates Note 1 to Theorem 2 in what concerns the choice of j for even m.

Case 2 ($m > 1$, $a > 0$). This case is reduced to Case 1 owing to the symmetry (Lemma 6).

Case 2.1 (k is odd, m is odd). According to Lemma 6, one should attach to the boundary lines II and III their symmetrical lines I and IV of Theorem 2.

Case 2.2 (k is even, m is odd). As in the last case, according to Lemma 6, to the boundary lines II and III one should attach their symmetrical lines I and IV of Theorem 2.

Case 2.3 (k is odd, m is even). Symmetry does not allow one to pass here from the results for the half-plane $a < 0$ to those for the half-plane $a > 0$. However, one can readily see that in the quadrant $a \geqslant 0$, $b \geqslant 0$ the inequality $a+b < 1$ is the necessary and sufficient condition for asymptotic stability of Eq. (1). Indeed, this condition is sufficient on the strength of the Cohn's result [9]. If it is violated, then $a + b \geqslant 1$, and it suffices to assume in (1) that $x_0 = x_1 = \ldots x_{k-1} = 1$ in order to establish that $x_n \geqslant 1$ for any n and, consequently, there is no asymptotic stability. Therefore, the line $a + b = 1$ (line I, Theorem 2) and, by virtue of Lemma 6, its symmetrical line III will be added to the boundaries II and IV of Case 1.3. Thus, Theorem 2 is proved for $m > 1$, $a \neq 0$. Additionally, direct look of Eq. (1) for $a = 0$ confirms correctness of boundaries I–IV of Theorem 2, which proves it for $m > 1$.

Case 3 ($m = 1$). We prove that the result of [6] is incorporated in the scheme of Theorem 2. According to Theorem 4, if one confines oneself to the half-plane $a \geqslant 0$, then for any—even or odd—k the boundaries of the stability domain of Eq. (5) are defined by the lines $a + b = 1$ (line I, Theorem 2) and

$$a = \sin k\omega/\sin(k-1)\omega, \quad b = -\sqrt{a^2 + 1 - 2a\cos\omega}, \tag{A.39}$$

where ω varies from 0 to π/k. By a trivial transformation of (A.39) with the replacement $\omega^* = \pi - \omega$, we obtain $a = -\sin k\omega^*/\sin(k-1)\omega^*, b = -(-1)^k \sin\omega^*/\sin(k-1)\omega^*$, where ω^* varies from $(k-1)\pi/k$ to π. We established boundary IV of Theorem 2. Boundaries III and II on the half-plane

$a < 0$ are obtained by symmetrical mapping of the respective boundaries I and IV according to Lemma 6, which proves Theorem 2.

Proof of Theorem 3. Let the theorem conditions be satisfied.

(1) By Theorem 2, in the quadrant Q_{00} the boundary of the asymptotic stability domain $D(k,m)$ is the same for all k and m—it is the straight line $a+b=1$. Therefore, $(k_1,m_1) \approx (k_2,m_2)$.

(2) Let k_1 and k_2 be odd, k_3 and k_4 be even.

(2.1) Let $k_1 < k_2$, $m_1 \leqslant m_2$ or $k_1 \leqslant k_2$, $m_1 < m_2$.

Case 1 ($m_2 > 1$). By Lemma 1, it suffices to prove for $(k_1,m_1) \succ (k_2,m_2)$ that $D(k_1,m_1) \bigcap Q_{10} \bigcap S \supset D(k_2,m_2) \bigcap Q_{10} \bigcap S$, where S is the band $(-1) < a < 0$. By virtue of (A.31) and (A.32), in the quadrant Q_{10} on the boundary II (Theorem 2) $b = u^*(\omega_j(a))$, where j is an even natural number satisfying condition (3) together with some even s. On the strength of (A.31), (A.26), and (A.27), by introducing in the notation the dependence on the variables on k, m, a we obtain that

$$b(k,m,a) = \sqrt{1 - a^2 + a^2 \cos^2 \beta(k,m,a)} + a \cos \beta(k,m,a), \tag{A.40}$$

where for any natural numbers k and m ($k > m$) the one-place function $\beta(k,m,a)$ is inverse to the one-place function (see (A.21))

$$a(k,m,\beta) = -\sin((k\beta - \pi)/(k-m))/\sin \beta. \tag{A.41}$$

The interval $(-1,0)$ is the domain of variation of the variable a in (A.40), (A.41). By assuming in (A.22) that $|p_j| = 1$, we see that the domain of variation of the variable β in (A.40), (A.41) is represented by the interval $(\pi/k, \pi/m)$ if $m > 1$ and the interval $(\pi/k, k\pi/(2k-1))$ if $m = 1$. The intersection of the limits of the functions $\beta(k_1,m_1,a)$ and $\beta(k_2,m_2,a)$ is the interval $(\pi/k_1, \pi/m_2)$ because we have $k_1\pi/(2k_1 - 1) > \pi/2 \geqslant \pi/m_2$ even for $m_1 = 1$. For $\pi/k_1 < \beta < \pi/m_2$,

$$0 < (k_1\beta - \pi)/(k_1 - m_1) < (k_2\beta - \pi)/(k_2 - m_2) < \pi/2. \tag{A.42}$$

We obtain from (A.41) and (A.42) that $a(k_1,m_1,\beta) > a(k_2,m_2,\beta)$. Therefore, in view of monotone decrease of $a(k,m,\beta)$, the inequality

$$\beta(k_1,m_1,a) > \beta(k_2,m_2,a) \tag{A.43}$$

is true for the inverse functions. Inequality (A.43) is proved for those values of the variable a for which the values of $\beta(k_1,m_1,a)$ and $\beta(k_2,m_2,a)$ hit the interval $(\pi/k_1, \pi/m_2)$. Outside this interval, inequality (A.43) is evident. It follows from (A.43) that $\cos\beta(k_1,m_1,a) < \cos\beta(k_2,m_2,a)$. Yet the function $y(x) = \sqrt{1 - a^2 + a^2x^2} + ax$ decreases monotonically over the interval $(-1,1)$ of value of x for $a \in (-1,0)$. Therefore, (see (A.40)) $b(k_1,m_1,a) > b(k_2,m_2,a)$ for any $a \in (-1,0)$, which means that $D(k_1,m_1) \bigcap Q_{10} \bigcap S \supset D(k_2,m_2) \bigcap Q_{10} \bigcap S$. Therefore, $(k_1,m_1) \succ (k_2,m_2)$.

Case 2 ($m_2 = 1$). Then, $m_1 = 1$ as well. By Theorem 4, the boundary of the domain $D(k,1)$ in Q_{10} is the line

$$b(k,a) = \sqrt{a^2 + 1 + 2a\cos\omega(k,a)}, \tag{A.44}$$

where $\omega(k,a)$ is the root of the equation $a = -\sin k\omega/\sin(k-1)\omega$ over the interval $(0,\pi/k)$ for any $a \in (-k/(k-1), 0)$.

Since $k_1 < k_2$, we get $-\sin k_1\omega/\sin(k_1-1)\omega < -\sin k_2\omega/\sin(k_2-1)\omega$ for $\omega \in (0,\pi/k_2)$. Therefore, $\omega(k_1,a) > \omega(k_2,a)$ and $b(k_1,a) > b(k_2,a)$ by virtue of (A.44). Hence, $(k_1,1) \succ (k_2,1)$ in Case 2. It goes without saying that m_3 and m_4 are odd owing to evenness of k_3 and k_4, and by Theorem 2, the line $-a+b=1$ is the boundary of the domains $D(k_3,m_3)$ and $D(k_4,m_4)$ in the quadrant Q_{10}, which results in $(k_1,m_1) \succ (k_2,m_2) \succ (k_3,m_3) \approx (k_4,m_4)$ and proves Item 2.1.

(2.2) On the boundary II of Theorem 2 (see also (A.35)) we calculate for $\omega = j\pi/k$ and $\omega = s\pi/m$ with regard for (3), respectively,

$$db/da = (-1)^j \cos(\pi/k) \quad \text{and} \quad db/da = (-1)^j/\cos(\pi/m) \tag{A.45}$$

(formulas (A.45) for $m = k - 1$ can be found in [2]). Let $k_1 > k_2$, $m_1 < m_2$. On the boundary II, j is even in Q_{10}, therefore, from (A.45) at the point $a = 0$, $b = 1$ we obtain that $\dfrac{db(k_1, m_1, a)}{da} > \dfrac{db(k_2, m_2, a)}{da}$, at the point $a = -1$, $b = 0$ we obtain $\dfrac{db(k_1, m_1, a)}{da} < \dfrac{db(k_2, m_2, a)}{da}$. Therefore, in some neighborhood $U(0, 1)$ of the point $a = 0$, $b = 1$ the inclusion $D(k_1, m_1) \bigcap Q_{10} \bigcap\ U(0, 1) \supset D(k_2, m_2) \bigcap Q_{10} \bigcap U(0, 1)$ occurs, and in some neighborhood $U(-1, 0)$ of the point $a = -1$, $b = 0$ the inclusion $D(k_2, m_2) \bigcap Q_{10} \bigcap U(-1, 0) \supset D(k_1, m_1) \bigcap Q_{10} \bigcap U(-1, 0)$ holds. Consequently, the pairs (k_1, m_1) and (k_2, m_2) are incomparable, which proves Item 2. Items 3 and 4 are proved along the same lines, which completes the proof of Theorem 3.

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