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A note on explicit stability conditions for autonomous higher order difference equations

M. M. KIPNIS†* and D. A. KOMISSAROVA‡

†Department of Mathematics, Chelyabinsk State Pedagogical University, Chelyabinsk 454080, Russia
‡Department of Mathematics, South Ural State University, Chelyabinsk 454080, Russia

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We prove that if $a_s \geq 0$ $(1 \leq s \leq k)$ and

$$0 < \sum_{s=1}^{k} \frac{a_s}{2\sin\frac{\pi}{2(2s-1)}} < 1,$$

then the equation

$$x(n) = x(n-1) - \sum_{s=1}^{k} a_s x(n-s)$$

is asymptotically stable. As a corollary, we obtain sufficient asymptotic stability conditions:

$$a_s \geqslant 0 \quad \text{and} \quad 0 < \sum_{s=1}^{k} s a_s \leq \frac{\pi}{2}$$

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1. Introduction and the main results

The following result is well known.

THEOREM 1.1 (SEE REF. [5]). *Equation* $x(n) = x(n-1) - ax(n-s)$, $s \in \mathbb{N}$, $a \in \mathbb{R}$, *is asymptotically stable if and only if*

$$0 < \frac{a}{2\sin\frac{\pi}{2(2s-1)}} < 1.$$

We extend the result in some direction. Let $k \in \mathbb{N}$, $a_s \in \mathbb{R}$ $(1 \leq s \leq k)$. Consider the equation

$$x(n) = x(n-1) - \sum_{s=1}^{k} a_s x(n-s). \tag{1}$$

*Corresponding author. Email:kipnis@mail.ru

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THEOREM 1.2. *If $a_s \geq 0$ $(1 \leq s \leq k)$ and*

$$0 < \sum_{s=1}^{k} \frac{a_s}{2\sin\frac{\pi}{2(2s-1)}} < 1,$$

then equation (1) *is asymptotically stable.*

We first prove an elementary technical lemma.

LEMMA 1.3. *For every $\omega \in (0, \pi]$, there exists $m \in \mathbb{R}$, such that for every $s \in \mathbb{N}$*

$$\frac{1}{\sin\frac{\pi}{2(2s-1)}} + \frac{\sin(m-s)\omega}{\sin\frac{\omega}{2}\cos\left(m-\frac{1}{2}\right)\omega} \geq 0. \tag{2}$$

Proof. For $\omega = \pi$, the proof is evident. Fix $\omega \in (0, \pi)$. Put

$$m = \frac{1}{2} + \frac{1}{\omega}\arctan\left(\frac{\omega}{\pi\tan\frac{\omega}{2}}\right).$$

Now equation (2) becomes

$$F(\omega) := \frac{\sin\frac{\omega}{2}}{\sin\frac{\pi}{2(2s-1)}} - \sin\left(s-\frac{1}{2}\right)\omega + \frac{\omega\cos\left(s-\frac{1}{2}\right)\omega}{\pi\tan\frac{\omega}{2}} \geq 0. \tag{3}$$

Case 1. $\omega \in (0, 2\pi/(2s-1)]$. Let f be given by

$$f(\omega) = \frac{\sin\frac{\omega}{2}}{\sin\frac{\pi}{2(2s-1)}\cos\left(s-\frac{1}{2}\right)\omega} - \tan\left(s-\frac{1}{2}\right)\omega + \frac{\omega}{\pi\tan\frac{\omega}{2}}, \tag{4}$$

so that $F(\omega) = f(\omega)\cos(s-(1/2))\omega$. f has a removable singularity at $\omega = \pi/(2s-1)$ and $\lim f(\omega) = 0$ when $\omega \to \pi/(2s-1)$. Put $f(\pi/(2s-1)) = 0$. The inequality (3) will be proved once we prove decreasing of f when $\omega \in (0, 2\pi/(2s-1)] \bigcap (0, \pi)$. The third constituent in equation (4) is a decreasing function. What is left is to show that the function

$$\varphi(\omega) = \frac{\sin\frac{\omega}{2}}{\sin\frac{\pi}{2(2s-1)}\cos\left(s-\frac{1}{2}\right)\omega} - \tan\left(s-\frac{1}{2}\right)\omega$$

(with the removable singularity at $\omega = \pi/(2s-1)$) decreasing when $\omega \in (0, 2\pi/(2s-1)] \bigcap (0, \pi)$. Indeed,

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\omega} = \frac{1}{\cos^2\left(s-\frac{1}{2}\right)\omega\sin\frac{\pi}{2(2s-1)}}\psi(\omega),$$

where

$$\psi(\omega) = \frac{1}{2}\cos\frac{\omega}{2}\cos\left(s-\frac{1}{2}\right)\omega + \left(s-\frac{1}{2}\right)\sin\frac{\omega}{2}\sin\left(s-\frac{1}{2}\right)\omega - \left(s-\frac{1}{2}\right)\sin\frac{\pi}{2(2s-1)}.$$

Nonpositivity of $\psi(\omega)$ guaranteed by $\psi(\pi/(2s-1)) = 0$ and

$$\frac{\mathrm{d}\psi}{\mathrm{d}\omega} = s(s-1)\sin\frac{\omega}{2}\cos\left(s-\frac{1}{2}\right)\omega.$$

Hence, if $\omega \in (0, \pi/(2s-1)]$, then $(\mathrm{d}\psi/d\omega) \geq 0$, while if $\omega \in [(\pi/(2s-1)), (2\pi/(2s-1))]$, then $(\mathrm{d}\psi/\mathrm{d}\omega) \leq 0$. For the Case 1, lemma is proved.

Case 2. $\omega > (2\pi/(2s-1))$. By $\omega < \pi$, we get $s > (3/2)$. Since $s \in \mathbb{N}$, we have $s \geq 2$. Then

$$\frac{\sin\frac{\omega}{2}}{\sin\frac{\pi}{2(2s-1)}} \geq \frac{\sin\frac{2\pi}{2(2s-1)}}{\sin\frac{\pi}{2(2s-1)}} \geq \frac{\sin\frac{2\pi}{6}}{\sin\frac{\pi}{6}} > 1.73,$$

which gives (see equation (3)) $F(\omega) > 1.73 - 1 - (2/\pi) > 0$.

Proof of Theorem 1.2. Characteristic equation for equation (1) is the following:

$$\lambda^k - \lambda^{k-1} + \sum_{s=1}^{k} a_s\lambda^{k-s} = 0. \tag{5}$$

The root of the form $\lambda = \mathrm{e}^{\mathrm{i}\omega}$ $(\omega \in [0, \pi])$ comes into being in equation (5) at the stability boundary in the space of the parameters $(a_1, \ldots, a_k)$. Hence, at the stability boundary there exists $\omega \in [0, \pi]$ such that

$$\cos k\omega - \cos(k-1)\omega + \sum_{s=1}^{k} a_s\cos(k-s)\omega = 0, \tag{6}$$

$$\sin k\omega - \sin(k-1)\omega + \sum\nolimits_{s=1}^{k} a_s\sin(k-s)\omega = 0. \tag{7}$$

If $a_1 = 0, \ldots, a_{k-1} = 0$, $a_k > 0$ and a_k is sufficiently small, then equation (1) is asymptotically stable [5]. Hence, it suffices to show that if $a_s \geq 0$ $(1 \leq s \leq k)$ and if the point $(a_1, \ldots, a_k)$ is not the origin of coordinates and if equations (6) and (7) hold for $\omega \in (0, \pi]$ then

$$\sum_{s=1}^{k} \frac{a_s}{2\sin\frac{\pi}{2(2s-1)}} \geq 1. \tag{8}$$

Indeed, let $a_s \geq 0$ $(1 \leq s \leq k)$ and $\omega \in (0, \pi]$ and equations (6) and (7) hold. Let m be given by Lemma 1.3. Multiplying equations (6) and (7) by

$$\frac{(-1)\sin(k-m)\omega}{2\sin\frac{\omega}{2}\cos\left(m-\frac{1}{2}\right)\omega} \quad \text{and} \quad \frac{\cos(k-m)\omega}{2\sin\frac{\omega}{2}\cos\left(m-\frac{1}{2}\right)\omega}$$

correspondingly and adding the obtained equalities, we get

$$1 + \sum_{s=1}^{k} a_s \frac{\sin(m-s)\omega}{2\sin\frac{\omega}{2}\cos\left(m-\frac{1}{2}\right)\omega} = 0. \tag{9}$$

By nonnegativity of a_s $(1 \le s \le k)$ from equations (2) and (9), we conclude

$$(-1) + \sum_{s=1}^{k} \frac{a_s}{2\sin\frac{\pi}{2(2s-1)}} = \sum_{s=1}^{k} a_s \left(\frac{1}{2\sin\frac{\pi}{2(2s-1)}} + \frac{\sin(m-s)\omega}{2\sin\frac{\omega}{2}\cos\left(m-\frac{1}{2}\right)\omega} \right) \ge 0,$$

and equation (8) is proved. □

The following sufficient conditions for the asymptotic stability of equation (1) were known before (see Refs. [1,2, Corollary 4.2]): $a_s \ge 0$ $(1 \le s \le k)$, $0 < \sum_{s=1}^{k} s a_s < 1 + (1/e)$. They are more restrictive than those provided by Theorem 1.2. Some related results can be found in Ref. [4].

Since

$$2\sin\frac{\pi}{2(2s-1)} > \frac{\pi}{2s}, \quad s \in \mathbb{N},$$

we obtain the following consequence of Theorem 1.2.

THEOREM 1.4. *If $a_s \ge 0$ $(1 \le s \le k)$ and $0 < \sum_{s=1}^{k} s a_s \le (\pi/2)$, then equation (1) is asymptotically stable.*

2. Discussion

Theorem 1.2 cannot be improved in some sense. The following theorem affirms this assertion.

THEOREM 2.1. *Suppose that there are positive numbers $A_1, \ldots, A_k$ such that for any nonnegative sequence (a_s), $(1 \le s \le k)$, the condition*

$$0 < \sum_{s=1}^{k} \frac{a_s}{A_s} < 1$$

implies that equation (1) is asymptotically stable. Then

$$A_s \le 2\sin\frac{\pi}{2(2s-1)} \quad (1 \le s \le k).$$

Theorem 2.1 can be easily proved from Theorem 1.1. Theorem 2.1 does not means that the inequality

$$\sum_{s=1}^{k} \frac{a_s}{2\sin\frac{\pi}{2(2s-1)}} < 1$$

is a necessary stability condition for equation (1). Generally, boundaries of the stability domain in the space of parameters of equation (1) are nonconvex [3].

The constant $\pi/2$ in Theorem 1.4 cannot be improved, too. Theorem 1.4 affirms the hypothesis proposed in Ref. [3]. The following theorem can be easily derived from Theorems 1.2 and 1.4.

THEOREM 2.2. *Assume that $a_s \geq 0$ for any $s \in \mathbb{N}$. If either*

$$0 < \sum_{s=1}^{\infty} \frac{a_s}{2\sin\frac{\pi}{2(2s-1)}} < 1$$

or

$$0 < \sum_{s=1}^{\infty} s a_s \leq \frac{\pi}{2}$$

then the Volterra difference equation $x(n) = x(n-1) - \sum_{s=1}^{n} a_s x(n-s)$ is asymptotically stable.

Theorems 1.4 and 2.2 are analogous to the following results about a differential and integrodifferential equations.

THEOREM 2.3 (SEE REF. [6], THEOREMS 1 AND 2 AND REF. [4], THEOREM 1).

1. *If $a_s > 0$, $\tau_s > 0$ $(1 \leq s \leq k)$ and $\sum_{s=1}^{k} a_s \tau_s < (\pi/2)$, then the equation $\dot{x}(t) = -\sum_{s=1}^{k} a_s x(t-\tau_s)$ is asymptotically stable.*
2. *Let $a(\tau): \mathbb{R} \to \mathbb{R}$. Let $a(\tau)$ be continuous and $a(\tau) > 0$ when $\tau > 0$. If $\int_0^{\infty} a(\tau)\,\mathrm{d}\tau < (\pi/2)$, then the equation $\dot{x}(t) = -\int_0^{\infty} a(\tau)x(t-\tau)\,\mathrm{d}\tau$ is asymptotically stable.*

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