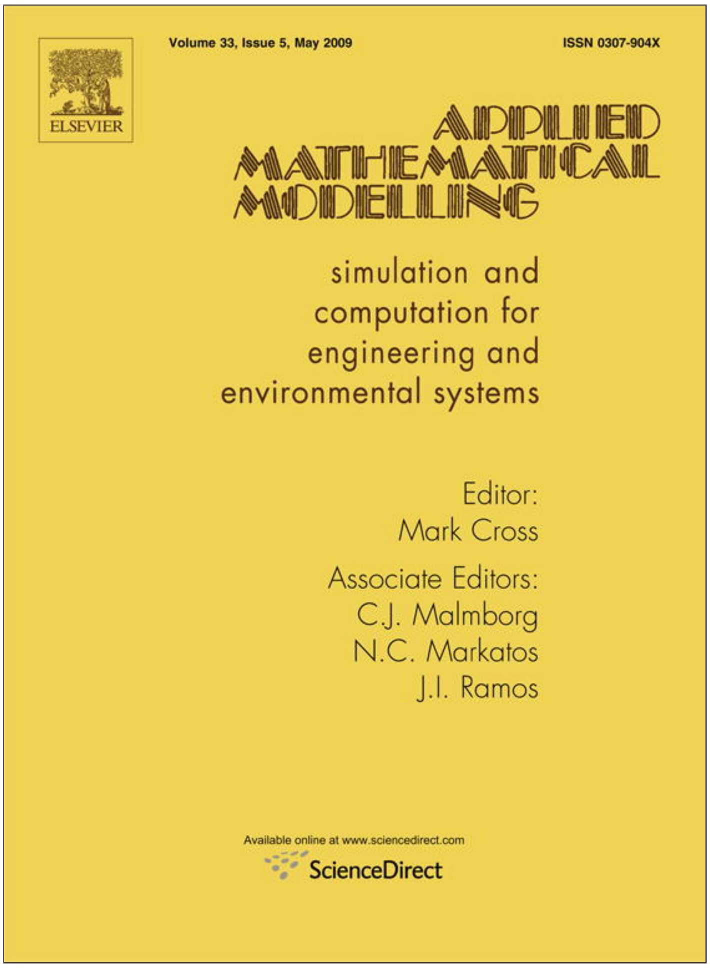

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Stability criteria for a nonlinear nonautonomous system with delays ☆

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ABSTRACT

This paper further develops a method, originally introduced by Mori et al., for proving local stability of steady states in linear systems of delay differential equations. A nonlinear nonautonomous system of delay differential equations with several delays is considered. Explicit delay-independent sufficient conditions for global attractivity of the solutions with an extremely simple form are provided. The above-mentioned conditions make the stability test quite practical. We illustrate application of this test to the Hopfield neural network models. The results obtained were also applied to a new marine protected areas model with delay that describes the ecological linkage between the reserve and fishing ground.

1. Introduction

The general theory of scalar autonomous delay differential equations (DDE's) has been well studied [1], whereas the theory of the systems of DDE's, especially nonautonomous systems, is a relatively new research endeavor [2–6].

Mori et al. [7] obtained a simple criterion for asymptotic stability of the solutions of the linear autonomous system with delay

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Ax(t) + Bx(t-\tau), \tag{1}$$

where A and B are constant matrices.

van den Driessche et al. [8] studied the dynamics of artificial networks in signal and image processing (Hopfield-type neural network) and obtained sufficient conditions for global attractivity of the solutions of the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Ax(t) + Bg(x(t-\tau)) + J, \tag{2}$$

where B is a constant matrix, J is a constant vector, matrix A is a diagonal matrix $A = \mathrm{diag}(a_1, a_2, \ldots, a_n)$, and function $g(u) = (g_1(u), g_2(u), \ldots, g_n(u))^{\mathrm{T}}$ is bounded and globally Lipschitz function, i.e., $|g(u) - g(v)| \leqslant L|u - v|$ for every pair (u, v).

Inspired by technique developed in [7], we consider the nonlinear nonautonomous system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Ax(t) + F(t, x(t-\tau_1), \ldots, x(t-\tau_k)), \quad t \geqslant 0,$$

$\tau_i = \mathrm{const} > 0$, where A is a constant Hurwitz $n \times n$ matrix, F maps $[0, \infty) \times \mathbb{R}^{m \times n}$ into $\mathbb{R}^n$.

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In this paper, we generalize the results obtained in [7,8] in several directions: (i) we address the stability of nonlinear systems, which was not done in [7], (ii) proved new results for nonautonomous systems, (iii) relax sufficient conditions obtained in [8] for the Hopfield networks and (iv) exploit sufficient conditions for a new population dynamics model of marine protected areas.

The organization of this paper is as follows. The basic concepts are presented in Section 2; in Section 3 we give explicit and delay-independent sufficient conditions for the global stability of the system. New sufficient conditions for the stability of the Hopfield neural networks are obtained in Section 3. Applications of our findings to a new model of marine protected areas are presented in Section 4.

2. Preliminaries

Consider the nonlinear system

$$\begin{aligned}&\frac{\mathrm{d}x}{\mathrm{d}t} = Ax(t) + F(t, x(t-\tau_1), \ldots, x(t-\tau_k)), \quad t > 0,\\ &0 < \tau_1 \leqslant \tau_2 \leqslant \ldots \leqslant \tau_k, \quad x(t) = \phi(t), \quad t \in [-\tau_k, 0],\end{aligned} \tag{3}$$

where $x : [-\tau_k, \infty] \to \mathbb{R}^n, A \in \mathbb{R}^{n\times n}$. Here $F : \mathbb{R}_+ \times \mathbb{R}^{n\times k} \to \mathbb{R}^n$ nonlinear and continuous vector function.

We shall use $|\cdot|, \|\cdot\|$ to denote norms in $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$. As usual,

$$|Ax| \leqslant \|A\| \cdot |x|.$$

The following matrix function plays an essential role in this paper.

Let $\vartheta(A) : \mathbb{R}^{n\times n} \to \mathbb{R}$ be a function such that

$$\|\mathrm{e}^{At}\| \leqslant \mathrm{e}^{\vartheta(A)t}. \tag{4}$$

A matrix measure $\mu(A)$, defined in [9] by

$$\mu(A) = \lim_{\varepsilon\to+0} \frac{\|I + \varepsilon A\| - 1}{\varepsilon}, \tag{5}$$

is an example of such a function. Matrix measures are often used [2,4,7,9] in robust stability analysis to obtain less conservative results than if norms are used.

Remark 1. Clearly, if

$$\|A\| = \max_j \sum_{i=1}^{n} |a_{ij}|, \tag{6}$$

then

$$\mu(A) = \max_{1\leqslant j\leqslant n} \left\{ a_{jj} + \sum_{k=1,k\neq j}^{k=n} |a_{kj}| \right\}. \tag{7}$$

If $\mu(A) < 0$ then matrix A is a Hurwitz matrix [9].

Mori et al. [7] proved that *zero solution $x(t)$ of linear system (1) is asymptotically stable, provided that $-\mu(A) > \|B\|$.*

3. Main results

To study stability of system (3) we make two basic assumptions:

Let there exist the nonnegative sequence $\gamma_i (1 \leqslant i \leqslant k)$, such that for every $u_i \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$ the inequalities

$$(H_1) \qquad |F(t, u_1, u_2, \ldots, u_k)| \leqslant \sum_{i=1}^{k} \gamma_i |u_i|,$$

$$(H_2) \qquad -\vartheta(A) > \sum_{i=1}^{k} \gamma_i,$$

hold.

Theorem 3.1. *Suppose that conditions (H_1) and (H_2) are satisfied. Then there are positive constants α and C such that every solution $x(t)$ of the system (3) satisfies the inequality*

$$|x(t)| \leqslant Ce^{-\alpha t} \max_{-\tau_k\leqslant t\leqslant 0} |\phi(t)|. \tag{8}$$

Proof. From Eq. (3) we have

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}F(t, x(s-\tau_1), \ldots, x(s-\tau_k))\,ds.$$

Thus,

$$|x(t)| \leqslant \|e^{At}\| \cdot |x(0)| + \int_0^t \|e^{A(t-s)}\| \cdot |F(x(s-\tau_1), \ldots, x(s-\tau_k))|\,ds$$

and

$$|x(t)| \leqslant \|e^{At}\| \cdot |x(0)| + \sum_{i=1}^{k} \gamma_i \int_0^t \|e^{A(t-s)}\| \cdot |x(s-\tau_i)|\,ds.$$

Notation $|x(t)| = v(t)$ yields

$$v(t) \leqslant e^{\vartheta(A)t}v(0) + \sum_{i=1}^{k} \gamma_i \int_0^t e^{\vartheta(A)(t-s)}v(s-\tau_i)\,ds, \tag{9}$$

for $t > 0$ and $v(t) = |\phi(t)|$ for $t \in [-\tau_k, 0]$.

Consider the scalar integral equation

$$z(t) = e^{\vartheta(A)t}v(0) + \sum_{i=1}^{k} \gamma_i \int_0^t e^{\vartheta(A)(t-s)}z(s-\tau_i)\,ds, \tag{10}$$

for $t > 0$ and $z(t) = |\phi(t)|$ for $t \in [-\tau_k, 0]$.

In order to prove Theorem 3.1 we need Lemma 3.1 as described below.

Lemma 3.1. *For the functions defined by (9) and (10) the inequality $z(t) \geqslant v(t)$ holds for all $t \geqslant 0$.*

Proof. To prove Lemma we shall use the method of steps. Let $u(t) = z(t) - v(t)$. Based on (9) and (10) clearly for $t > 0$

$$u(t) \geqslant \sum_{i=1}^{k} \gamma_i \int_0^t e^{\vartheta(A)(t-s)}u(s-\tau_i)\,ds, \tag{11}$$

and $u(t) = 0$ for $t \in [-\tau_k, 0]$.

For $t \in [0, \tau_1]$ the integral in the right hand side of (11) is equal to zero, thus $u(t) \geqslant 0$. The latter proves that the integral in (11) is nonnegative for $t \in [\tau_1, 2\tau_1]$. The process may be continued to prove Lemma 3.1. Eq. (10) is equivalent to the scalar differential equation below

$$\begin{aligned} \frac{dz}{dt} &= \vartheta(A)z(t) + \sum_{i=1}^{k} \gamma_i z(t-\tau_i), \quad t > 0, \\ z(t) &= |\phi(t)| \quad \text{for} \quad t \in [-\tau_k, 0]. \quad \square \end{aligned} \tag{12}$$

We need the following results.

Lemma 3.2. [1]. *If $\gamma_i \in \mathbb{R}$ $(1 \leqslant i \leqslant k)$, and a function $\vartheta(A)$ satisfies (H_2), then there are positive constants α and C such that every solution $z(t)$ of scalar Eq. (12) satisfies the inequality*

$$|z(t)| \leqslant Ce^{-\alpha t} \max_{-\tau_k \leqslant t \leqslant 0} |\phi(t)|.$$

Finally, by virtue of Lemmas 3.1, 3.2 we get

$$|x(t)| = v(t) \leqslant z(t) \leqslant Ce^{-\alpha t} \max_{-\tau_k \leqslant t \leqslant 0} |\phi(t)|.$$

The proof of Theorem 3.1 is complete. □

Corollary 3.1. *Let u_0 be an equilibrium of the Hopfield model*

$$\frac{du}{dt} = Au(t) + Bg(u(t-\tau)) + J, \tag{13}$$

where A and B are constant matrices and J is a constant vector.

Assume the following:

(T) Function g is globally Lipschitz with Lipschitz constant L.

If $-\vartheta(A) > L\|B\|$ then u_0 is a unique global attractor for system (13).

Proof. Let $x = u - u_0$ then

$$\frac{dx}{dt} = Ax(t) + F(x(t - \tau)),$$

where $F(x) = B[g(x + u_0) - g(u_0)]$. Condition (T) yields

$$|F(x)| \leqslant \|B\| \cdot |g(x + u_0) - g(u_0)| \leqslant L|x|\|B\|.$$

Therefore all conditions of Theorem 3.1 are satisfied and $\lim_{t\to\infty} x(t) = 0$. □

Remark 2. Two of the conditions of Theorems 2.1–2.4 in [8] are relaxed by Corollary 3.1, i.e. matrix A be diagonal and function g be bounded.

4. Global stability of a new marine protected area model

To illustrate biological applications of Theorem 3.1, we consider two habitat areas, with a fish population dispersing between the two areas, whilst fishing takes place only in region 2, with region 1 established as a no-fishing zone. To describe the ecological linkage between the reserve and fishing ground we propose the following nonautonomous nonlinear system of DDE's

$$\begin{aligned} \frac{dx_1}{dt} &= -[m_1 + D_1]x_1(t) + D_2x_2(t) + \gamma_1 x_1(t - \tau)\exp(-\alpha_1 x_1(t - \tau)), \\ \frac{dx_2}{dt} &= -[m_2 + D_2 + E]x_2(t) + D_1x_1(t) + \gamma_2 x_2(t - \tau)\exp(-\alpha_2 x_2(t - \tau)). \end{aligned} \tag{14}$$

Here, $m_i > 0$ – natural mortality death rates, $D_i > 0$ – dispersal rates, $E > 0$ – harvesting rate; $\tau \geqslant 0$ – maturation time, i.e., the time to develop from newborns to reproductively active adults, α_i and γ_i are nonnegative constants. The initial conditions are given by

$$x_i(t) = \varphi_i(t), t \in [-\tau, 0], \quad (i = 1, 2), \tag{15}$$

where due to the obvious ecological reason,

$$\varphi_i(t) \geqslant 0, \quad \text{for} \quad t \in [-\tau, 0]. \tag{16}$$

We will use a vector form of system (14)

$$\frac{dx}{dt} = Ax + f(x(t - \tau)), \tag{17}$$

where

$$A = \begin{bmatrix} -(m_1 + D_1) & D_2 \\ D_1 & -(m_2 + D_2 + E) \end{bmatrix}, \quad x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad f(x(t - \tau)) = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

with

$$f_i = \gamma_i x_i(t - \tau)\exp(-\alpha_i x_i(t - \tau)), \quad (i = 1, 2).$$

Theorem 4.1. *If*

$$\min(m_1, m_2 + E) > \max(\gamma_1, \gamma_2), \tag{18}$$

then zero solution of the system (14) is globally asymptotically stable.

Proof. Firstly, note that from standard differential equation theory [1], system (14) has solution $x_1(t) > 0, x_2(t) > 0$ for $t > 0$, provided that $x_1(0) > 0, x_2(0) > 0$ and

$$x_1(t) = \varphi_1(t) \geqslant 0, \quad x_2(t) = \varphi_2(t) \geqslant 0, \tag{19}$$

for $t \in [-\tau, 0]$.

Define the matrix norm by (6). Then for small $\varepsilon > 0$ we have

$$\|I + \varepsilon A\| = \left\| \begin{matrix} 1 - \varepsilon(m_1 + D_1) & \varepsilon D_2 \\ \varepsilon D_1 & 1 - \varepsilon(m_2 + D_2 + E) \end{matrix} \right\| = \max\{1 - \varepsilon m_1, 1 - \varepsilon(m_2 + E)\} = 1 - \varepsilon \min\{m_1, m_2 + E\}. \tag{20}$$

Therefore,

$$\mu(A) = -\min\{m_1, m_2 + E\}.$$

To prove global attractivity of the solution of Eq. (14), it is suffices that condition (H_1) of Theorem 3.1 holds for any vector $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ with nonnegative coordinates. Therefore,

$$|f(u)| = \left|\mathrm{diag}\{\gamma_1, \gamma_2\} \cdot \begin{bmatrix} u_1 e^{-\alpha_1 u_1} \\ u_2 e^{-\alpha_2 u_2} \end{bmatrix}\right| \leqslant \|\mathrm{diag}\{\gamma_1, \gamma_2\}\| \cdot |u| = \max\{\gamma_1, \gamma_2\}|u|. \tag{21}$$

Thus by Theorem 3.1 a zero solution of system (14) is asymptotically globally stable. □

We would like to point out that assumption (18) is biologically motivated: to avoid population extinction under the excessive harvesting, the minimum “loss” in every region should be less than the maximum “gain” in the region.

5. Discussion

The important feature of the present paper is extension of the technique developed for linear delay systems to the nonlinear nonautonomous systems with multiple delays. The still widely used method is the approach of Lyapunov-like functionals. Not by using any Lyapunov stability theorems, we established the sufficient conditions for the stability of the system.

To illustrate practicality of the stability test obtained, we studied two different applications: the Hopfield network model and the marine protected areas model.

Discussions for global asymptotic stability of the Hopfield network models allow a nonlinear term g in (13) to be unbounded and a constant matrix A not to be diagonal which are then more general than those of [8]; e.g., our results should be applicable to the networks models that are not necessarily symmetric. A similar methodology can be carried out to the networked control systems with time delays and nonlinear perturbation.

Via the system of nonlinear nonautonomous differential equations with delay, we introduced a new model for marine protected areas. For this fishery model we obtained sufficient conditions for a fish population extinction, the latter, from the point of view of fishery managers, is necessary for planning harvesting strategies and sustaining the fishing grounds.

Our study shows that an interesting but challenging problem associated with system (3) is the local and global stability of the nontrivial solutions of system (3). Another interesting modification of system (3) would be the incorporation of the time-varying and/or delay-dependent matrix A in the system. We leave these to future investigation.

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